A Study on Solving the Boundary Value Problem of Three-Region Composite Modified Bessel Equation

Xiao-xu Dong, Zhi-bin Liu , Shun-chu Li

***Abstract* This paper studies the boundary value problem (BVP) of three-region composite modified Bessel equation. Firstly, on the basis of similar structure of solution of BVP of differential equation, a new method for solving the class of BVPs is put forward and its steps are summed up. Secondly, the flow chart of algorithm of the new method is given and a program which corresponds with it is compiled. Finally, the new method is applied to solving a given BVP of three-region composite modified Bessel equation and the curve of solution of the boundary value problem is drawn by running the program on the computer. The new method is simple and direct to solve a class of boundary value problems.**

***Keywords* BVP; Three-region composite modified Bessel equation; Similar kernel function; Function of guide solution**

I. Introduction

At the beginning of this century, the thought of similar structure of the solution of the BVP of differential equation began to form in Ref 1. Some gratifying results have been achieved.

Based on the similar structure of solution of BVP of differential equation, Refs.2 and Ref.5 simplified solutions of BVPs of a class of second-order linear homogeneous differential equations and a class of second-order partial differential equations. It can be a good help to understand inherent laws of analytic solution. Furthermore, the unification between the similarity of solution and of the corresponding numbers and shapes has harmoniously improved the triad of mathematical theory.

With reducing the analysis expression of fixed solution problem on a class of composite Bessel equation and composite modified Bessel equation, Ref.3 and Ref.4 gained the solution's formal similarity. This similar structure of solution explained that solutions of the class equations can be given by the product of several fractions and the graphs have similarity too.

By analyzing the solution of the reservoir pressure and dimensionless reservoir pressure distribution in Laplace space, which was aimed at the well test analytic model of composite reservoir, Ref.6 discovered the similar structure of the solution's form of the composite reservoir under three kinds of outer boundary conditions (infinite, constant pressure and closed). Furthermore, this paper made the theoretical graph and analyzed the influence of the well-bore storage and skin effect on dimensionless reservoir pressure and dimensionless bottom-hole pressure by using numerical method of inversion.

Ref.7 studied the similar structure of solution in the Laplace space for the class of composite parabolic partial differential equations with the infinite outer boundary condition and the kind of inner boundary condition in connection with time t. The work can be a good help to understand inherent laws of relevant engineering science and design practical analysis software.

Aiming at the boundary value problem of homogeneous linear second-order ordinary differential equations(systems) and the mixed problem for homogeneous linear second-order partial differential equations(systems), Ref.8 reviewed some results of preliminary explorations with regard to the similar structure theory of their solutions or their Laplace space solutions and analyzed the formation idea of the similar solution and its significance of research and took apart the relation between the similar structure of its solution and governing equation and the boundary conditions, it also found a way to construct the solution of definite problems by making use of the formula of similar structure together with the similar kernel function and introduces the existing achievements which have been applied to permeation fluid mechanics.

The percolation model of composite reservoir was established in Ref.9, which considers the effective well-bore radius and well-bore storage. By using Laplace transform, the exact solutions of reservoir pressure and bottom-hole pressure were obtained under different boundary conditions in Laplace space. According to the theory of similar structure of solution, Ref.9 defined the kernel functions which are only relevant to two linear independent solutions of governing equations and outer boundary conditions, what's more, different outer boundary conditions correspond to different similar kernel functions. There is a similar structure among the solutions under three different outer boundaries.

On the basis of the analysis of the so1ution for a BVP of the linear homogeneous second-order differential equation,Ref.10 studied the similar structure of solution and similar kernel functions, and put forward a new method of solving the class of BVPs——the similar constructive method. the method is an innovative idea and a simple effective method of solving the boundary value problem of the differential equation .

The BVP of two points was considered for second-order linear homogeneous differential equation in Ref.11. The existence and uniqueness of solution were proved. The similar structure and similar kernel function of the solution were found .

Manuscript received March 15, 2016

Xiao-xu Dong, College of Science, Southwest Petroleum University, Chengdu, 610500, China
Zhi-bin Liu, College of Science, Southwest Petroleum University, Chengdu, 610500, China
Shun-chu Li, College of Science, Xihua University, Chengdu, 610039, China

Based on the analysis of the solution of a BVP of composite second-order linear homogeneous differential equation, Ref.12 studied similar kernel functions and the similar structure of the solution and put forward a new method for solving the class of boundary value problems.

Ref.13 solved a class of BVPs of the composite first Weber system. Solution with a form of continued fraction product to BVP of the composite first Weber system was obtained. Then a new method was obtained for solving the composite BVP—Similar Constructing Method. The method is simple and effective for solving the complicated BVP of differential system.

On the basis of similar structure of solution of the second-order linear homogeneous differential equation' BVP, Ref.14 proposed a new simple solution —similar constructive method of solution and summed up its detailed steps. A mathematical model of fractal reservoir with spherical flow under three kinds of outer boundary conditions (infinite, constant pressure and closed) was set up, in which influences of skin factor and well-bore storage are taken into consideration. And then SCMS is applied to solve it.

In this paper, the BVP of three-region composite modified Bessel equation is studied. Knowledge which is related to study of this paper is given In part I. In part II, on the basis of similar structure of solution of BVP of differential equation, a new method for solving the class of BVPs is put forward and its steps are summed up. In part III, the flow chart of algorithm of the new method is given and a program which corresponds with it is compiled. In part IV, the new method is applied to solving a given BVP of three-region composite modified Bessel equation and the curve of solution of the BVP is drawn by running the program on the computer.

II. Solution of the BVP

In this paper, the following BVP of three-region composite modified Bessel equation is studied:

$$\begin{cases} x^2 y_1'' + x y_1' - \left(x^2+\nu_1^2\right) y_1 = 0, & (a \le x \le b) \\ x^2 y_2'' + x y_2' - \left(x^2+\nu_2^2\right) y_2 = 0, & (b \le x \le c) \\ x^2 y_3'' + x y_3' - \left(x^2+\nu_3^2\right) y_3 = 0, & (c \le x \le d) \\ \left[E y_1 + (1+EF) y_1'\right]_{x=a} = D \\ y_1|_{x=b} = \lambda_1\, y_2|_{x=b},\ y_1'|_{x=b} = \lambda_2\, y_2'|_{x=b} \\ y_2|_{x=c} = \mu_1\, y_3|_{x=c},\ y_2'|_{x=c} = \mu_2\, y_3'|_{x=c} \\ \left[M y_3 + N y_3'\right]_{x=d} = 0 \end{cases} \tag{1.1}$$

Here D, E, F, M, N, a, b, c, d, λ_1, λ_2, μ_1, μ_2, ν_1, ν_2, ν_3 are constants and $\lambda_1\lambda_2\mu_1\mu_2 \ne 0$, $M^2+N^2 \ne 0$ and $D \ne 0$.

$I_{\nu_i}(x)$ and $K_{\nu_i}(x)$ are two linear independent solutions of second-order linear differential equations $x^2 y_i'' + x y_i' - \left(x^2+\nu_i^2\right) y_i = 0\,(x>0, i=1,2,3)$, here $I_n(\cdot)$, $K_n(\cdot)$ are respectively the first and the second class of modified Bessel functions of order n [15]. Defining a binary function is as below:

$$\varphi_{m,n}(x,\xi) = I_m(x)K_n(\xi) + (-1)^{m-n+1} K_m(x) I_{n_j}(\xi) \tag{1.2}$$

We lead into functions of guide solution as follows:

$$\varphi_{0,0}^{i}(x,\xi) \triangleq \varphi_{\nu_i,\nu_i}(x,\xi) \quad (i=1,2,3) \tag{1.3}$$

$$\varphi_{1,0}^{i}(x,\xi) \triangleq \frac{\partial}{\partial x}\varphi_{\nu_i,\nu_i}(x,\xi) = \frac{\nu_i}{x}\varphi_{\nu_i,\nu_i}(x,\xi) + \varphi_{\nu_i+1,\nu_i}(x,\xi) \tag{1.4}$$

$$\varphi_{0,1}^{i}(x,\xi) \triangleq \frac{\partial}{\partial \xi}\varphi_{\nu_i,\nu_i}(x,\xi) = \frac{\nu_i}{\xi}\varphi_{\nu_i,\nu_i}(x,\xi) - \varphi_{\nu_i,\nu_i+1}(x,\xi) \tag{1.5}$$

$$\begin{aligned}\varphi_{1,1}^{i}(x,\xi) &\triangleq \frac{\partial^2}{\partial x \partial \xi}\varphi_{\nu_i,\nu_i}(x,\xi) \\ &= \frac{\nu_i^2}{x\xi}\varphi_{\nu_i,\nu_i}(x,\xi) + \frac{\nu_i}{\xi}\varphi_{\nu_i+1,\nu_i}(x,\xi) - \frac{\nu_i}{x}\varphi_{\nu_i,\nu_i+1}(x,\xi) - \varphi_{\nu_i+1,\nu_i+1}(x,\xi)\end{aligned} \tag{1.6}$$

Where $i=1$ denotes inner region $(a \le x \le b)$, $i=2$ denotes middle region $(b \le x \le c)$, $i=3$ denotes outer region $(c \le x \le d)$.

Theorem If the boundary value problem (1.1) has unique solution, then solutions of inner, middle and outer regions are expressed respectively as follows (the detailed proof process is presented in Appendix A):

$$y_1(x) = D \cdot \frac{1}{E + \dfrac{1}{F+\Phi_1(a)}} \cdot \frac{1}{F+\Phi_1(a)} \cdot \Phi_1(x) \quad (a \le x \le b) \tag{2.1}$$

$$y_2(x) = D \cdot \frac{1}{E + \dfrac{1}{F+\Phi_1(a)}} \cdot \frac{1}{F+\Phi_1(a)} \cdot \frac{\varphi_{0,1}^{1}(b,b)}{\lambda_1 \Phi_2(b)\varphi_{1,1}^{1}(a,b) - \lambda_2 \varphi_{1,0}^{1}(a,b)} \cdot \Phi_2(x) \quad (b \le x \le c) \tag{2.2}$$

$$\begin{aligned} y_3(x) = D \cdot \frac{1}{E + \dfrac{1}{F+\Phi_1(a)}} \cdot \frac{1}{F+\Phi_1(a)} \\ \times \frac{\varphi_{0,1}^{1}(b,b)\varphi_{0,1}^{2}(c,c)}{\left[\lambda_1 \Phi_2(b)\varphi_{1,1}^{1}(a,b) - \lambda_2 \varphi_{1,0}^{1}(a,b)\right]\left[\mu_1 \Phi_3(c)\varphi_{1,1}^{2}(b,c) - \mu_2 \varphi_{1,0}^{2}(b,c)\right]} \cdot \Phi_3(x) \quad (c \le x \le d) \end{aligned} \tag{2.3}$$

Where $\Phi_3(x)$, $\Phi_2(x)$ and $\Phi_1(x)$ are called similar kernel functions of outer, middle and inner regions respectively and they are expressed as follows:

$$\Phi_3(x) = \frac{M\varphi_{0,0}^{3}(x,d) + N\varphi_{0,1}^{3}(x,d)}{M\varphi_{1,0}^{3}(c,d) + N\varphi_{1,1}^{3}(c,d)} \quad (c \le x \le d) \tag{2.4}$$

$$\Phi_2(x) = \frac{\mu_2\varphi_{0,0}^{2}(x,c) - \mu_1\Phi_3(c)\varphi_{0,1}^{2}(x,c)}{\mu_2\varphi_{1,0}^{2}(b,c) - \mu_1\Phi_3(c)\varphi_{1,1}^{2}(b,c)} \quad (b \le x \le c) \tag{2.5}$$

$$\Phi_1(x) = \frac{\lambda_2\varphi_{0,0}^{1}(x,b) - \lambda_1\Phi_2(b)\varphi_{0,1}^{1}(x,b)}{\lambda_2\varphi_{1,0}^{1}(a,b) - \lambda_1\Phi_2(b)\varphi_{1,1}^{1}(a,b)} \quad (a \le x \le b) \tag{2.6}$$

III. STEPS AND FLOWCHART OF ALGORITHM FOR SOLVING THE BVP

3.1 Steps for solving the BVP

According to solution procedures of the above boundary value problem, a new method for solving the boundary value problem of three-region composite modified Bessel equation is obtained. Specific steps are as follows:

Step1. Solving governing equations

Two linear independent solutions $I_{v_i}(x), K_{v_i}(x)$ $(i=1,2,3)$ of governing equations are obtained respectively by solving governing equations of inner, middle and outer regions of the boundary value problem (1.1).

Step2. Constructing functions of guide solution

Functions of guide solution of inner, middle and outer regions $\varphi_{0,0}^{i}(x,\xi)(i=1,2,3)$ are constructed respectively by using $I_{v_i}(x), K_{v_i}(x)$ $(i=1,2,3)$, as shown Eq,(1.3). Other functions of guide solution can be obtained by calculating partial derivatives of $\varphi_{0,0}^{i}(x,\xi)(i=1,2,3)$ to x, ξ respectively, as shown Eqs.(1.4)-(1.6).

Step3. Constructing similar kernel functions of inner, middle and outer regions

Firstly, the similar kernel function $\Phi_3(x)$ of outer region of the boundary value problem (1.1) can be structured by using functions of guide solution of outer region and coefficients M, N of the homogeneous outer boundary condition, as shown Eq.(2.4). Further we calculate $\Phi_3(c)$. Secondly, the similar kernel function $\Phi_2(x)$ of middle region can be structured by using functions of guide solution of middle region, coefficients μ_1, μ_2 of two connection conditions of middle and outer region and $\Phi_3(c)$, as shown Eq.(2.5). Further we calculate $\Phi_2(b)$. Finally, the similar kernel function $\Phi_1(x)$ of inner region of the boundary value problem (1.1) can be structured by using functions of guide solution of inner region, coefficients λ_1, λ_2 of two connection conditions of inner and middle regions and $\Phi_2(c)$, as shown Eq.(2.6). Further we calculate $\Phi_1(a)$.

Step4. Obtaining the solution of the boundary value problem

According to Eq.(2.1), Eq.(2.2) and Eq.(2.3), solutions of the inner, middle and outer regions are obtained respectively by assembling coefficients D, E, F of the non-homogeneous inner boundary condition, similar kernel functions $\Phi_1(x)$, $\Phi_2(x)$, $\Phi_3(x)$, values of $\Phi_1(a), \Phi_2(b), \Phi_3(c)$, functions of guide solution of inner and middle regions, coefficients λ_1, λ_2, μ_1, μ_2 of four convergence conditions of three regions.

3.2 The flow chart of algorithm of solving steps

According to steps of the above method, a flow chart (Fig. 1) of the algorithm of above method is drawn as follows:

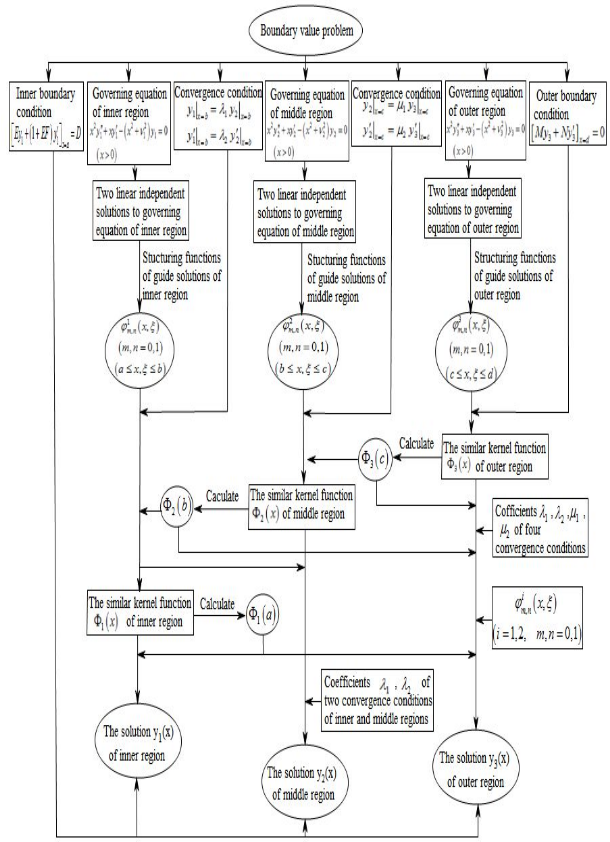

Fig.1 The flow chart of algorithm of the method

It clearly presents the relationship between solutions of three regions and similar kernel functions, functions of guide solution, boundary conditions and connection conditions. It elaborately illustrates the solution procedure for the boundary value problem of three-region composite modified Bessel equation.

IV. THE APPLICATION OF THE OBTAINED STEPS

The following boundary value problem of three-region composite modified Bessel is solved:

$$\begin{cases} x^2y_1''+xy_1'-x^2y_1=0, \quad (1\le x\le 4) \\ x^2y_2''+xy_2'-\left(x^2+1\right)y_2=0, \quad (4\le x\le 8) \\ x^2y_3''+xy_3'-\left(x^2+4\right)y_3=0, \quad (8\le x\le 12) \\ 2y_1|_{x=1}+7y_1'|_{x=1}=3 \\ y_1|_{x=4}=y_2|_{x=4}, \ y_1'|_{x=4}=2y_2'|_{x=4} \\ y_2|_{x=8}=y_3|_{x=8}, \ y_2'|_{x=8}=2y_3'|_{x=8} \\ 4y_3|_{x=12}+y_3'|_{x=12}=0 \end{cases} \tag{4.1}$$

Comparing with the boundary value problem (1.1) and (4.1), it can conclude $v_1=0$, $v_2=1$, $v_3=2$, $a=1$, $b=2$, $c=3$, $d=4$, $\lambda_1=1$, $\lambda_2=2$, $\mu_1=1$, $\mu_2=2$, $D=3$, $E=2$, $F=3$, $M=4$, $N=1$. The boundary value problem (4.1) has unique solution (the detailed proof process is presented in Appendix B). According to steps of the new method, the solution procedures for the boundary value problem (4.1) are as follows:

Step1. Solving governing equations

Two linear independent solutions $I_{v_i}(x)$, $K_{v_i}(x)$ $(i=1,2,3)$ [15] of governing equations are obtained respectively by solving governing equations of inner, middle and outer regions of the boundary value problem (4.1).

Step2. Constructing functions of guide solution

According to Eqs.(1.2)-(1.5), functions of guide solution of inner, middle and outer regions are constructed by using two linear independent solutions $I_{v_i}(x)$, $K_{v_i}(x)$ $(i=1,2,3)$ of governing equations of inner, middle and outer regions of the boundary value problem (4.1) respectively as follows:

$$\varphi_{0,0}^{1}(x,\xi)=I_0(x)K_0(\xi)-K_0(x)I_0(\xi)$$

$$\varphi_{1,0}^{1}(x,\xi)=I_1(x)K_0(\xi)+K_1(x)I_0(\xi)$$

$$\varphi_{0,1}^{1}(x,\xi)=-I_0(x)K_1(\xi)-K_0(x)I_1(\xi)$$

$$\varphi_{1,1}^{1}(x,\xi)=-I_1(x)K_1(\xi)+K_1(x)I_1(\xi)$$

$$\varphi_{0,0}^{2}(x,\xi)=I_1(x)K_1(\xi)-K_1(x)I_1(\xi)$$

$$\varphi_{1,0}^{2}(x,\xi)=\frac{1}{x}\left[I_1(x)K_1(\xi)-K_1(x)I_1(\xi)\right]+I_2(x)K_1(\xi)+K_2(x)I_1(\xi)$$

$$\varphi_{0,1}^{2}(x,\xi)=\frac{1}{\xi}\left[I_1(x)K_1(\xi)-K_1(x)I_1(\xi)\right]-I_1(x)K_2(\xi)-K_1(x)I_2(\xi)$$

$$\varphi_{1,1}^{2}(x,\xi)=\frac{1}{x\xi}\left[I_1(x)K_1(\xi)-K_1(x)I_1(\xi)\right]+\frac{1}{\xi}\left[I_2(x)K_1(\xi)+K_2(x)I_1(\xi)\right]-\frac{1}{x}\left[I_1(x)K_2(\xi)+K_1(x)I_2(\xi)\right]-\left[I_2(x)K_2(\xi)-K_2(x)I_2(\xi)\right]$$

$$\varphi_{0,0}^{3}(x,\xi)=I_2(x)K_2(\xi)-K_2(x)I_2(\xi)$$

$$\varphi_{1,0}^{3}(x,\xi)=\frac{2}{x}\left[I_2(x)K_2(\xi)-K_2(x)I_2(\xi)\right]+I_3(x)K_2(\xi)+K_3(x)I_2(\xi)$$

$$\varphi_{0,1}^{3}(x,\xi)=\frac{2}{\xi}\left[I_2(x)K_2(\xi)-K_2(x)I_2(\xi)\right]-I_2(x)K_3(\xi)-K_2(x)I_3(\xi)$$

$$\varphi_{1,1}^{3}(x,\xi)=\frac{4}{x\xi}\left[I_2(x)K_2(\xi)-K_2(x)I_2(\xi)\right]+\frac{2}{\xi}\left[I_3(x)K_2(\xi)+K_3(x)I_2(\xi)\right]-\frac{2}{x}\left[I_2(x)K_3(\xi)+K_2(x)I_3(\xi)\right]-\left[I_3(x)K_3(\xi)-K_3(x)I_3(\xi)\right]$$

Step3. Constructing similar kernel functions of inner, middle and outer regions

According to Eqs.(2.4)-(2.6), similar kernel functions of outer, middle and inner region of the boundary value problem (4.1) are constructed as follows:

$$\Phi_3(x)=\frac{4\varphi_{0,0}^{3}(x,12)+\varphi_{0,1}^{3}(x,12)}{4\varphi_{1,0}^{3}(8,12)+\varphi_{1,1}^{3}(8,12)}\quad(8\le x\le 12)$$

$$\Phi_2(x)=\frac{2\varphi_{0,0}^{2}(x,8)-\Phi_3(8)\varphi_{0,1}^{2}(x,8)}{2\varphi_{1,0}^{2}(4,8)-\Phi_3(8)\varphi_{1,1}^{2}(4,8)}\quad(4\le x\le 8)$$

$$\Phi_1(x)=\frac{2\varphi_{0,0}^{1}(x,4)-\Phi_2(4)\varphi_{0,1}^{1}(x,4)}{2\varphi_{1,0}^{1}(1,4)-\Phi_2(4)\varphi_{1,1}^{1}(1,4)}\quad(1\le x\le 4)$$

Thus

$$\Phi_3(8)=\frac{4\varphi_{0,0}^{3}(8,12)+\varphi_{0,1}^{3}(8,12)}{4\varphi_{1,0}^{3}(8,12)+\varphi_{1,1}^{3}(8,12)}$$

$$\Phi_2(4)=\frac{2\varphi_{0,0}^{2}(4,8)-\Phi_3(8)\varphi_{0,1}^{2}(4,8)}{2\varphi_{1,0}^{2}(4,8)-\Phi_3(8)\varphi_{1,1}^{2}(4,8)}$$

$$\Phi_1(1)=\frac{2\varphi_{0,0}^{1}(1,4)-\Phi_2(4)\varphi_{0,1}^{1}(1,4)}{2\varphi_{1,0}^{1}(1,4)-\Phi_2(4)\varphi_{1,1}^{1}(1,4)}$$

Step4. Obtaining the solution of the boundary value problem (4.1)

According to Eqs.(2.1)- (2.3), solutions of inner, middle and outer regions of the boundary value problem (4.1) can be obtained respectively as follows:

$$y_1(x)=\frac{3}{7+2\cdot\Phi_1(1)}\cdot\Phi_1(x)\quad(1\le x\le 4)$$

$$y_2(x)=\frac{3}{7+2\cdot\Phi_1(1)}\cdot\frac{\varphi_{0,1}^{1}(4,4)}{\Phi_2(4)\varphi_{1,1}^{1}(1,4)-2\varphi_{1,0}^{1}(1,4)}\cdot\Phi_2(x)\quad(4\le x\le 8)$$

$$y_3(x)=\frac{3}{7+2\cdot\Phi_1(1)}\cdot\frac{\varphi_{0,1}^{1}(4,4)\varphi_{0,1}^{2}(8,8)}{\left[\Phi_2(4)\varphi_{1,1}^{1}(1,4)-2\varphi_{1,0}^{1}(1,4)\right]\left[\Phi_3(8)\varphi_{1,1}^{2}(4,8)-\varphi_{1,0}^{2}(4,8)\right]}\cdot\Phi_3(x)\quad(8\le x\le 12)$$

According to above steps, a corresponding program is compiled. Then a curve (Fig. 2) of solutions of three regions of the boundary value problem (4.1) is drawn out as below:

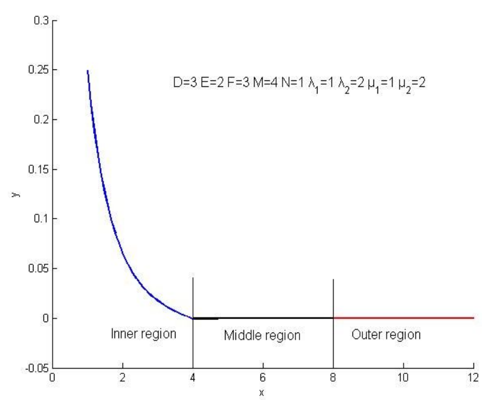

Fig.2 The curve of solution of the boundary value problem (4.1)

V. COMPARISON ANALYSIS

The mathematical model of three-region composite reservoir which considers the impact of formation pressure and skin factor in Laplace space is as below:

$$\begin{cases} \frac{d^2\overline{P}_{1D}}{dr_D^2}+\frac{1}{r_D}\cdot\frac{d\overline{P}_{1D}}{dr_D}=z\overline{P}_{1D}, \quad 1\le r_D\le\alpha \\ \frac{d^2\overline{P}_{2D}}{dr_D^2}+\frac{1}{r_D}\cdot\frac{d\overline{P}_{2D}}{dr_D}=\sigma z\overline{P}_{2D}, \quad r_D\ge\alpha \\ \overline{P}_{\omega D}(z)=\left[\overline{P}_{1D}-Sr_D\frac{d\overline{P}_{1D}}{dr_D}\right]\Big|_{r_D=1} \\ \left(r_D\frac{d\overline{P}_{1D}}{dr_D}\right)\Big|_{r_D=1}=-\left(\frac{1}{z}-C_Dz\overline{P}_{\omega D}\right) \\ \overline{P}_{1D}(\alpha,z)=\overline{P}_{2D}(\alpha,z),\ \frac{d\overline{P}_{1D}}{dr_D}\Big|_{r_D=\alpha}=\lambda\frac{d\overline{P}_{2D}}{dr_D}\Big|_{r_D=\alpha} \\ \overline{P}_{2D}(\infty,z)=0 \end{cases} \tag{14}$$

Next, we solve the boundary value problem by using different methods and compare the different methods.

First, we solve the boundary value problem (14) by using the method of the literature [10]. We only obtain the result as below:

$$\begin{cases} I_0(\sqrt{\xi})A+K_0(\sqrt{\xi})B-\overline{P}_{\omega D}=0 \\ \sqrt{\xi}I_1\left(\sqrt{z/C_De^{2S}}\right)A-\sqrt{\xi}K_1(\sqrt{\xi})B-z\overline{P}_{\omega D}=0 \\ I_0(\sqrt{\xi}r_D)A+K_0(\sqrt{\xi}r_D)B-I_0(\sqrt{\xi\sigma}r_D)C-K_0(\sqrt{\xi\sigma}r_D)D=0 \\ \sqrt{\xi}I_1(\sqrt{\xi}r_D)A-\sqrt{\xi}K_1(\sqrt{\xi}r_D)B-\lambda\sqrt{\xi\sigma}I_1(\sqrt{\xi\sigma}r_D)C+\lambda\sqrt{\xi\sigma}K_1(\sqrt{\xi\sigma}r_D)D=0 \\ \lim_{r_D\to\infty}\left[I_0(\sqrt{\xi\sigma}r_D)C+K_0(\sqrt{\xi\sigma}r_D)D\right]=0 \end{cases} \tag{15}$$

Here $\xi=z/C_De^{2S}$ .Systems (15) includes 5 equations and 5 unknown number ($A,B,C,D,\overline{P}_{\omega D}$), so the Laplace space solution of the bottom hole function $\overline{P}_{\omega D}$ is acquired by using the Gauss iterative method for solving the systems (15).

Second, we solve the boundary value problem (14) by using the method of the literature [11]. We obtain the result as below:

$$\overline{P}_{\omega D}=\frac{1}{z}\cdot\frac{temp1\cdot temp0+temp2}{C_Dz(temp1\cdot temp0+temp2)-(temp3\cdot temp0+temp4)} \tag{16}$$

Here

$$temp1=I_0(\sqrt{z})-\frac{1}{2}S\sqrt{z}\left(I_{-1}(\sqrt{z})+I_1(\sqrt{z})\right)$$

$$temp2=K_0(\sqrt{z})+\frac{1}{2}S\sqrt{z}\left(K_{-1}(\sqrt{z})+K_1(\sqrt{z})\right)$$

$$temp3=\frac{1}{2}\sqrt{z}\left(I_{-1}(\sqrt{z})+I_1(\sqrt{z})\right)$$

$$temp4=-\frac{1}{2}\sqrt{z}\left(K_{-1}(\sqrt{z})+K_1(\sqrt{z})\right)$$

$$temp0=\frac{-\alpha\sqrt{\sigma z}tempK_0(\alpha\sqrt{z})+\frac{1}{2}\alpha\sqrt{z}\left(K_{-1}(\alpha\sqrt{z})+K_1(\alpha\sqrt{z})\right)}{\alpha\sqrt{\sigma z}tempI_0(\alpha\sqrt{z})+\frac{1}{2}\alpha\sqrt{z}\left(I_{-1}(\alpha\sqrt{z})+I_1(\alpha\sqrt{z})\right)}$$

$$temp=\frac{K_1(\alpha\sqrt{\sigma z})}{K_0(\alpha\sqrt{\sigma z})}$$

Third, the method of the literature [12] is numerical method, so we can't obtain the analytical solution of the boundary value problem (14).

Final, we solve the boundary value problem (14) by using the method of this paper.

$$p_{wD}(z)=\frac{1}{z}\cdot\frac{1}{C_Dz-\frac{1}{\Phi_1(1,z)-S}} \tag{17}$$

Here,

$$\Phi_{11}(r_D,z)=\frac{\Phi_{21}(\alpha,z)\sqrt{z}\Psi_{0,1}(r_D,\alpha,\sqrt{z})+\lambda\Psi_{0,0}(r_D,\alpha,\sqrt{z})}{\Phi_{21}(\alpha,z)z\Psi_{1,1}(1,\alpha,\sqrt{z})+\lambda\sqrt{z}\Psi_{1,0}(1,\alpha,\sqrt{z})}$$

$$\Phi_{21}(r_D,z)=-\frac{K_0(\sqrt{\sigma z}r_D)}{\sqrt{\sigma z}K_1(\alpha\sqrt{\sigma z})}$$

$$\Psi_{m,n}(\alpha,\beta,y)=I_m(\alpha y)K_n(\beta y)+(-1)^{m-n+1}K_m(\alpha y)I_n(\beta y)$$

By comparing the above results, we can obtain the result as Tab. 1:

Literature	Result	Form of solution	Deviation	Computation complexity	Time complexity
[10]	Expression of the solution isn't given	Analytical solution	No	High	High
[11]	Equation (16)	Analytical solution	Yes	High	Low
[12]	Expression of the solution	numerical solution	No	High	High

	isn't given				
This paper	Equation (17)	Analytical solution	No	Low	Low

Tab. 1 Comparison of several methods

According to Tab. 1, we can see that the method of this paper is better than the other three.

CONCLUSIONS

(1) When dealing with the boundary value problem of three-region composite modified Bessel equation, only two linear independent solutions $I_{v_i}(x), K_{v_i}(x)\ (i = 1,2,3)$ of governing equations of inner, middle and outer regions respectively are obtained. Then the boundary value problem can be solved by using steps of the above new method.

(2) According to steps of the new method, it is clear that the method is a convenient, effective and creative way to solve the boundary value problem of three-region composite modified Bessel equation.

(3) Similar structures of solution of inner, middle and outer regions clearly show the relationship between solutions of inner, middle and outer regions and similar kernel functions, functions of guide solution that is generated by using two linear independent solutions of governing equations, boundary conditions and connection conditions.

(4) According to steps of the above new method, a corresponding program is compiled. It is applied to drawing a graph of the solution of the boundary value problem (4.1). And the graph clearly illustrates the solution of the boundary value problem.

(5) Many existing seepage fluid models of composite reservoir can be adapted to the boundary value problem of the composite modified Bessel equation. So we can use the new method to solve the problem of seepage fluid models of composite reservoir. And the new method is proposed to solve a class of complicated problems in reality.

REFERENCES

1. Poon D. International Meeting of the Pet . Soc. of. CIM / SPE9-534, 1995.
2. Fikri J, Kuchuk, Tarek M, Habashy. Solution of Pressure Diffusion in Radially Composite Reservoirs [J]. Transport in Porous Media. 19(1995): 199-232.
3. Abdurrahman Satman. An Analytical Study of Transient Flow in Stratified Systems with Fluid Banks [A. SPE 10264, 1997.
4. X. Peng, B. Yu, X. Qiao, S. Qiu, and Z. Jiang, "Radial permeability of fractured porous media by Monte Carlo simulations," International Journal of Heat and Mass Transfer, vol. 57, no. 1, pp. 369–374, 2013.
5. P. Xu, S. Qiu, B. Yu, and Z. Jiang, "Prediction of
6. relative permeability in unsaturated porous media with a fractal approach," International Journal of Heat and Mass Transfer, vol. 64, pp 829–837, 2013.
7. M. C. Santizo, "A semi-analytic solution for flow in finite conductivity vertical fractures using fractal theory," in SPE Annual Technical Conference and Exhibition, pp. 1–18, 2012, 163057-STU.
8. M. Cossio, "A semi-analytic solution for flow in finite-conductivity vertical fractures using fractal theory," in Proceeding of the SPE Annual Technical Conference and Exhibition, San Antonio, Tex, USA, October 2012.
9. EZULDCE O, LGBOKOYI A. Horizontal well pressure transient analysis in anisotropic composite reservoirs-A three dimensional semi-analytical approach [J]. Journal of Petroleum Science and Engineering, 2012, 96(5):120-139.
10. Prob Thararoop, Zuleima T. Karpyn, Turgay Ertekin. Development of a multi-mechanistic, dual-porosity, dual-permeability, numerical flow model for coalbed methane reservoirs [J]. Journal of Natural Gas Science and Engineering 8 (2012):121-131
11. Li Shunchu, Jia Minhui. The Formal Similarity of Solutions on the Class of Differential Equation [J]. Journal of UEST of China, 2004, 33(Supp.):95-98.
12. JIA Minhui, LI Shunchu. The Similar Structure of Solution Differential Equation on Boundary Value Problem (in Chinese) [J]. College Mathematics, 2005, 21(5):37-39.
13. CHEN Zi-chun, LIU Peng-hui, LI Shun-chu. The Similar Structure of Composite Bessel Equation on Fixed Solution Problem (in Chinese)[J]. Journal of Chongqing Techno Business University (Natural Science Edition), 2006, 23(1):1-4(to 24).
14. LIU Peng-hui, CHEN Zi-chun, LI Shun-chu. Similar Structure of Fixed Solution Problems for Composite Abnormal Bessel Equations (in Chinese) [J]. Journal of Xihua University (Natural Science Edition), 2006, 25(2):23-26.
15. LI Shun-chu. The Formal Similarity of Solutions to the Class of 2-order Partial Differential Equation in the Laplace Space (in English) [J]. Journal of Xihua University (Natural Science Edition), 2007, 26(4):83-86.
16. LI Shun-chu, ZHENG Peng-she, ZHANG Yu-fei. The Similar Structure of Pressure Distribution in the Composite Reservoir (in Chinese) [J]. Journal of Mathematics in Practice and Theory, 2008, 38(3):23-28.
17. SU Jian-peng, LI Shun-chu, LI Cheng-jie. The Similar of Solutions in the Laplace Space of Composite Parabolic Partial Differential Equation [J]. Journal of Zaozhuang University, 2009, 26(2):6-11.
18. LI Shun-chu. Preliminary Exploration and Prospects of the Similar Structure of Solutions of Differential Equations (in Chinese) [J]. Journal of Xihua University (Natural Science Edition), 2010, 29(2):223-226(to238).
19. Li Quanyong, Li Shunchu, Li Wei, Tang Yibin. Study on percolation model of composite media reservoir based on similar structure of solution (in Chinese) [J]. Fault-Block Oil & Gas Field, 2011, 18(5): 623-625.
20. Li Shunchu, Liao Zhijian. Constructing the Solution of Boundary Value Problem of the Differential Equation with its an Arbitrary Non-trivial Solution (in Chinese) [J]. Journal of Sichuan University (Natural Science Edition), 2012, 49(6):1209-1213.
21. Li Shun-chu, Wu Xiao-qing. Several Important Properties for the Boundary Value Problems of Second-order Linear Homogeneous Differential Equation (in Chinese)[J].

Journal of Xihua University (Natural Science Edition), 2013, 32(1): 23-26.

22. Li Shunchu. The Similarity Structuring Method of Boundary Value Problems for Composite Differential Equations [J]. Journal of Xihua University (Natural science Edition), 2013, 32(4): 27-31.
23. Dong Xiaoxu, Li Shunchu, Gui Dongdong, Pu Jun, Li Huichun. Similar Constructing Method for Solving the Boundary Value Problem of the Composite First Weber equation [J]. American Journal of Applied Mathematics and Statistics, 2013, 1(4):76-82.
24. Cui-Cui Sheng, Jin-Zhou Zhao, Yong-Ming Li, Shun-Chu Li and Hu Jia. Similar Construction Method of Solution for Solving the Mathematical Model of Fractal Reservoir with Spherical Flow [J]. Journal of Applied Mathematics, vol. 2013, Article ID 219218, 8 pages, 2013.
25. Liu Shishi, Liu Shida. Special Function [M]. Beijing: China Meteorological Press, 2002.

A. *Appendix A*

B. *The Detailed Proof of Theorem*

In this section, the detailed proof of theorem is discussed.

$I_{v_i}(x)$ and $K_{v_i}(x)$ are two linear independent solutions of second-order linear homogeneous differential equations

$$x^2 y_i'' + x y_i' - \left(x^2 + v_i^2\right) y_i = 0, \quad (x > 0) \quad (i = 1,2,3).$$

It is universally known that general solutions of governing equations of inter, middle and outer regions of the boundary value problem (1.1) are [15]

$$y_i(x) = A_i I_{v_i}(x) + B_i K_{v_i}(x) \qquad (i = 1,2,3)$$

(A.1)

By substituting Eq.(A.1) into inner and outer boundary conditions and four convergence conditions of the boundary value problem (1.1), the following equations are obtained respectively:

$$\left\{ E I_{v_1}(a) + (1+EF)\left[\frac{v_1}{a} I_{v_1}(a) + I_{v_1+1}(a)\right]\right\} A_1 + \left\{ E K_{v_1}(a) + (1+EF)\left[\frac{v_1}{a} K_{v_1}(a) - K_{v_1+1}(a)\right]\right\} B_1 = D$$

(A.2)

$$I_{v_1}(b) A_1 + K_{v_1}(b) B_1 - \lambda_1 I_{v_2}(b) A_2 - \lambda_1 K_{v_2}(b) B_2 = 0$$

(A.3)

$$\left[\frac{v_1}{b} I_{v_1}(b) + I_{v_1+1}(b)\right] A_1 + \left[\frac{v_1}{b} K_{v_1}(b) - K_{v_1+1}(b)\right] B_1 - \lambda_2 \left[\frac{v_2}{b} I_{v_2}(b) + I_{v_2+1}(b)\right] A_2 - \lambda_2 \left[\frac{v_2}{b} K_{v_2}(b) - K_{v_2+1}(b)\right] B_2 = 0$$

(A.4)

$$I_{v_2}(c) A_2 + K_{v_2}(c) B_2 - \mu_1 I_{v_3}(c) A_3 - \mu_1 K_{v_3}(c) B_3 = 0$$

(A.5)

$$\left[\frac{v_2}{c} I_{v_2}(c) + I_{v_2+1}(c)\right] A_2 + \left[\frac{v_2}{c} K_{v_2}(c) - K_{v_2+1}(c)\right] B_2 - \mu_2 \left[\frac{v_3}{c} I_{v_3}(c) + I_{v_3+1}(c)\right] A_3 - \mu_2 \left[\frac{v_3}{c} K_{v_3}(c) - K_{v_3+1}(c)\right] B_3 = 0$$

(A.6)

$$\left\{ M I_{v_3}(d) + N\left[\frac{v_3}{d} I_{v_3}(d) + I_{v_3+1}(d)\right]\right\} A_3 + \left\{ M K_{v_3}(d) + N\left[\frac{v_3}{d} K_{v_3}(d) - K_{v_3+1}(d)\right]\right\} B_3 = 0$$

(A.7)

According to the existence and uniqueness of solution of the boundary value problem (1.1), it can be found that the coefficient determinant Δ of linear equations (Eqs.(A.2) - (A.7)) about undetermined coefficients is not equal to zero, and

$$\begin{aligned}\Delta = E\Big[&\lambda_2\mu_2 M \varphi_{0,0}^1(a,b)\varphi_{1,0}^2(b,c)\varphi_{1,0}^3(c,d) - M\lambda_2\mu_1\varphi_{0,0}^1(a,b)\varphi_{1,1}^2(b,c)\varphi_{0,0}^3(c,d)\\ &-\lambda_1\mu_2 M\varphi_{0,1}^1(a,b)\varphi_{0,0}^2(b,c)\varphi_{1,0}^3(c,d) + \lambda_1\mu_1 M\varphi_{0,1}^1(a,b)\varphi_{0,1}^2(b,c)\varphi_{0,0}^3(c,d)\\ &+\lambda_2\mu_2 N\varphi_{0,0}^1(a,b)\varphi_{1,0}^2(b,c)\varphi_{1,1}^3(c,d) - \lambda_2\mu_1 N\varphi_{0,0}^1(a,b)\varphi_{1,1}^2(b,c)\varphi_{0,1}^3(c,d)\\ &-\lambda_1\mu_2 N\varphi_{0,1}^1(a,b)\varphi_{0,0}^2(b,c)\varphi_{1,1}^3(c,d) + \lambda_1\mu_1 N\varphi_{0,1}^1(a,b)\varphi_{0,1}^2(b,c)\varphi_{0,1}^3(c,d)\Big]\\ +(1+EF)\Big[&\lambda_2\mu_2 M\,\varphi_{1,0}^1(a,b)\varphi_{1,0}^2(b,c)\varphi_{1,0}^3(c,d) - \lambda_2\mu_1 M\varphi_{1,0}^1(a,b)\varphi_{1,1}^2(b,c)\\ &\cdot\varphi_{0,0}^3(c,d) - \lambda_1\mu_2 M\varphi_{1,1}^1(a,b)\varphi_{0,0}^2(b,c)\varphi_{1,0}^3(c,d) + \lambda_1\mu_1 M\varphi_{1,1}^1(a,b)\varphi_{0,1}^2(b,c)\\ &\varphi_{0,0}^3(c,d) + \lambda_2\mu_2 N\varphi_{1,0}^1(a,b)\varphi_{1,0}^2(b,c)\varphi_{1,1}^3(c,d) - \lambda_2\mu_1 N\varphi_{1,0}^1(a,b)\varphi_{1,1}^2(b,c)\\ &\cdot\varphi_{0,1}^3(c,d) - \lambda_1\mu_2 N\varphi_{1,1}^1(a,b)\varphi_{0,0}^2(b,c)\varphi_{1,1}^3(c,d) + \lambda_1\mu_1 N\varphi_{1,1}^1(a,b)\varphi_{0,1}^2(b,c)\\ &\varphi_{0,1}^3(c,d)\Big]\end{aligned}$$

(A.8)

Values of $A_1, B_1, A_2, B_2, A_3, B_3$ can be obtained by application of the Cramer rule as follows:

$$A_1 = D\left\{K_{v_1}(b) - \left[\frac{v_1}{b} K_{v_1}(b) - K_{v_1+1}(b)\right]\Phi_2(b)\right\} \times \Big[E\varphi_{0,0}^1(a,b) - E\varphi_{0,0}^1(a,b)\Phi_2(b) + (1+EF)\varphi_{1,0}^1(a,b) - (1+EF)\varphi_{1,1}^1(a,b)\Phi_2(b)\Big]^{-1}$$

(A.9)

$$B_1 = -D\left\{I_{v_1}(b) - \left[\frac{v_1}{b} I_{v_1}(b) + I_{v_1+1}(b)\right]\Phi_2(b)\right\} \times \Big[E\varphi_{0,0}^1(a,b) - E\varphi_{0,1}^1(a,b)\Phi_2(b) + (1+EF)\varphi_{1,0}^1(a,b) - (1+EF)\varphi_{1,1}^1(a,b)\Phi_2(b)\Big]^{-1}$$

(A.10)

$$\begin{aligned}A_2 = \frac{D}{\Delta}\Big\{&-M\mu_2 K_{v_2}(c)\varphi_{0,1}^1(b,b)\varphi_{1,0}^3(c,d) + M\mu_1\left[\frac{v_2}{c} K_{v_2}(c) - K_{v_2+1}(c)\right]\varphi_{0,1}^1(b,b)\\ &\cdot\varphi_{0,0}^3(c,d) - N\mu_2 K_{v_2}(c)\varphi_{0,1}^1(b,b)\varphi_{1,1}^3(c,d) + N\mu_1\left[\frac{v_2}{c} K_{v_2}(c) - K_{v_2+1}(c)\right]\\ &\cdot\varphi_{0,1}^1(b,b)\varphi_{0,1}^3(c,d)\Big\}\end{aligned}$$

(A.11)

$$\begin{aligned}B_2 = -\frac{D}{\Delta}\Big\{&-M\mu_2 I_{v_2}(c)\varphi_{0,1}^1(b,b)\varphi_{1,0}^3(c,d) + M\mu_1\left[\frac{v_2}{c} I_{v_2}(c) + I_{v_2+1}(c)\right]\varphi_{0,1}^1(b,b)\\ &\cdot\varphi_{0,0}^3(c,d) - N\mu_2 I_{v_2}(c)\varphi_{0,1}^1(b,b)\varphi_{1,1}^3(c,d) + N\mu_1\left[\frac{v_2}{c} I_{v_2}(c) + I_{v_2+1}(c)\right]\\ &\cdot\varphi_{0,1}^1(b,b)\varphi_{0,1}^3(c,d)\end{aligned}$$

(A.12)

$$A_3 = \frac{D}{\Delta}\left\{MK_{v_3}(d) + N\left[\frac{v_3}{d}K_{v_3}(d) - K_{v_3+1}(d)\right]\right\}\varphi_{0,1}^1(b,b)\varphi_{0,1}^2(c,c)$$

(A.13)

$$B_3 = -\frac{D}{\Delta}\left\{MI_{v_3}(d) + N\left[\frac{v_3}{d}I_{v_3}(d) + I_{v_3+1}(d)\right]\right\}\varphi_{0,1}^1(b,b)\varphi_{0,1}^2(c,c)$$

(A.14)

By substituting values of $A_1, B_1, A_2, B_2, A_3, B_3$ (Eqs.(A.8)-(A.14)) into Eq.(A.1) and using the similar kernel function of outer region Eq.(2.4), the similar kernel function of middle region Eq.(2.5) and the similar kernel function of inner region Eq.(2.6), solutions of inner, middle and outer regions of the boundary value problem (1.1) are obtained respectively, i.e. Eq.(2.1), Eq.(2.2) and Eq.(2.3).

Appendix B

In this section, the boundary value problem (4.1) has unique solution, which is proved.

$I_0(x)$ and $K_0(x)$ are two linear independent solutions governing equation of inner region $x^2y_1'' + xy_1' - (x^2)y_1 = 0$. $I_1(x)$ and $K_1(x)$ are two linear independent solutions governing equation of middle region $x^2y_2'' + xy_2' - (x^2+1)y_2 = 0$. $I_2(x)$ and $K_2(x)$ are two linear independent solutions governing equation of outer region $x^2y_3'' + xy_3' - (x^2+4)y_3 = 0$. According to Appendix A, the corresponding coefficient matrix of linear equations about undetermined coefficients $A_1, B_1, A_2, B_2, A_3, B_3$ is

$$C = \begin{pmatrix} 2I_0(1)+7I_1(1) & 2K_0(1)-7K_1(1) & 0 & 0 & 0 & 0 \\ I_0(4) & K_0(4) & -I_1(4) & -K_1(4) & 0 & 0 \\ I_1(4) & -K_1(4) & -\frac{1}{2}I_1(4)-2I_2(4) & -\frac{1}{2}K_1(4)+2K_2(4) & 0 & 0 \\ 0 & 0 & I_1(8) & K_1(8) & -I_2(8) & -K_2(8) \\ 0 & 0 & \frac{1}{8}I_1(8)+I_2(8) & \frac{1}{8}K_1(8)-K_2(8) & -\frac{1}{2}I_2(8)-2I_3(8) & -\frac{1}{2}K_2(8)+2K_3(8) \\ 0 & 0 & 0 & 0 & \frac{25}{6}I_2(12)+I_3(12) & \frac{25}{6}K_2(12)-K_3(12) \end{pmatrix}$$

The row simplest form of matrix C is below :

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It is clear that $R(C) = 6 =$ numbers of undetermined coefficients , so the boundary value problem of (4.1) has unique solution.